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1982 J. Phys. A: Math. Gen. 15 L513

(http://iopscience.iop.org/0305-4470/15/9/014)

J. Phys. A: Math. Gen. 15 (1982) L513–L520. Printed in Great Britain

LETTER TO THE EDITOR

Parisi solutions for the m-vector spin glass in a field

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Received 21 May 1982

Abstract. An m-vector classical spin glass in a field is studied at the mean-field level in terms of a Parisi replica symmetry breaking scheme. Solutions for the mean-field equations are presented which have important consequences for the irreversibility of the spin glass phase and clearly indicate the relevance of reduced temperatures scaling as H^2, H, $H^{2/3}$.

Currently it is widely accepted that the onset of irreversibility at a spin glass transition may be interpreted in terms of spontaneous breaking of replica symmetry in a mean-field theory in which physical disorder is mapped into an interaction between replicated spins in an effective pure system. For the m-vector model in a field Cragg *et al* (1982) have explicitly demonstrated that the replica symmetric solution (Gabay and Toulouse 1981, to be referred to as GT) is thermodynamically unstable throughout the whole spin glass phase $T<T_{\mathrm{GT}}$ ($T_{\mathrm{GT}}(0)-T_{\mathrm{GT}}(H)\propto H^2$ for small H) but have speculated that longitudinal irreversibility may be weak until a substantially lower temperature, $(T_{\mathrm{GT}}(0)-T)\propto H^{2/3}$ for small H. Recent experiments (Chamberlain *et al* 1982) support the view that significant longitudinal irreversibility sets in at a temperature with the $H^{2/3}$ scaling. In order to understand the origin of these features better we present a replica symmetry broken description for the problem, based on the work of Parisi (1979a, b) for the Ising case. We give explicit solutions around the isotropic critical point which demonstrate the relevance of the various ranges of the reduced temperature $t=(T_{\mathrm{GT}}(0)-T)\sim H^2, H, H^{2/3}$ and vindicate the speculations of Cragg *et al.*

For simplicity we base our mean-field analysis on a Sherrington–Kirkpatrick (1975) model

$$\mathscr{H}=-\sum_{i,j} J_{ij}\boldsymbol{S}_i\cdot\boldsymbol{S}_j-H\sum_i S_{i1} \tag{1}$$

where the spins are classical m-dimensional vectors with $|\boldsymbol{S}|^2=m$ and the J_{ij} are quenched independently random exchanges of infinite range distributed with mean zero and variance $J/\sqrt{N}$. The field H is chosen in the Cartesian $\mu=1$ direction.

Using the replica trick we follow GT and obtain the free energy per spin in the thermodynamic limit in the form

$$f=-T\lim_{n\to0} n^{-1}\max\{F(p^{\alpha\beta},q^{\alpha\beta},x^{\alpha})\} \tag{2a}$$

where the free energy functional to be maximised is given by

$$F(q^{\alpha\beta}, p^{\alpha\beta}, x^{\alpha}) = -\tfrac{1}{2}\beta^2\Big(\sum_{(\alpha\beta)}(m-1)(q^{\alpha\beta})^2 + (p^{\alpha\beta})^2\Big) - \tfrac{1}{2}\beta^2\Big(\sum_{\alpha} m(m-1)(x^{\alpha})^2 + mx^{\alpha}\Big)$$
$$+\ln\Big\{\operatorname*{Tr}_{\{S^\alpha\}} \exp\Big[\beta^2 \sum_{(\alpha\beta)}\Big(q^{\alpha\beta}\sum_{\lambda\neq 1}^{m} S_\lambda^\alpha S_\lambda^\beta + p^{\alpha\beta} S_1^\alpha S_1^\beta\Big)$$
$$+\tfrac{1}{2}\beta^2 \sum_{\alpha} mx^{\alpha}(S_1^{\alpha})^2 + \beta H \sum_{\alpha} S_1^{\alpha}\Big]\Big\}. \tag{2b}$$

Here the superscripts α, $\beta = 1, \ldots, n$ are replica labels, the notation $(\alpha\beta)$ denoting a pair of *different* labels. For convenience, units have been chosen such that $J = k_B = 1$.

The replica symmetric (RS) solution has

$$q^{\alpha\beta} = q \overset{\lambda\neq 1}{=} \overline{\langle S_\lambda\rangle^2} = 1 - T\chi_T - x \qquad \text{all } \alpha, \beta \tag{3}$$
$$p^{\alpha\beta} = p = \overline{\langle S_1\rangle^2} = 1 - T\lambda_L + (m-1)x \qquad \alpha \neq \beta$$
$$x^{\alpha} = x = (\overline{\langle S_1^2\rangle} - 1)/(m-1)$$

where $\langle\rangle$ and $\overline{}$ denote respectively thermal and disorder averages and χ_L^l, χ_T^l the longitudinal and transverse *local* differential susceptibilities. For $H \neq 0$ p and x are everywhere non-zero while q has a transition from zero (paramagnet) to non-zero (transverse spin glass ordering) at

$$T_{GT} = 1 - (m^2 + 4m + 2)H^2/4(m+2)^2 + \text{higher orders in } H. \tag{4}$$

As Cragg *et al* have shown, for $T < T_{GT}$ the RS solution is, however, no longer stable.

Within the Ising model a scheme to deal with RS breaking has been proposed by Parisi (1979a, b). We now present an analogous scheme for the present problem near the isotropic critical point $H = 0$, $T = 1$ (for a related scheme with $H = 0$ and including axial anisotropy see Elderfield and Sherrington (1982)). In Parisi's theory p and q are replaced by real functions $p(r)$, $q(r)$ on the interval $(0, 1)$. The *thermodynamic* differential local susceptibilities are given by

$$T\chi_L^l = 1 + T^2\Big((m-1)x - \int_0^1 \mathrm{d}r\, p(r)\Big) \qquad T\chi_T^l = 1 - T^2\Big(x + \int_0^1 \mathrm{d}r\, q(r)\Big) \tag{5}$$

while

$$\overline{\langle S_1\rangle^2} = T^2 \max p(r) \equiv T^2 p_m \qquad \overline{\langle S_\mu\rangle^2} \overset{\mu\neq 1}{=} T^2 \max q(r) \equiv T^2 q_m. \tag{6}$$

The substitution of (6) into the final parts of the first two lines of (3) yields the corresponding *linear response* susceptibilities. The difference between the thermodynamic and linear response susceptibilities is a measure of irreversibility.

Within the Parisi analysis

$$f = -T \min\{F(p(r), q(r), x)\}$$

where

$$F(q(r), p(r), x)$$
$$= \frac{1}{2}h^2 - \frac{1}{4}\frac{h^4}{(m+2)} + \frac{1}{4}\frac{m(m-1)}{(m+2)} - \frac{1}{2}h^2\int_0^1 \mathrm{d}r\, p(r)$$

$$
\begin{aligned}
&+\frac{1}{4}(m-1)\left(\tau+\frac{2h^2}{(m+2)}\right)\int_0^1 \mathrm{d}r\, q^2(r) \\
&+\frac{1}{4}\left[\tau-2h^2\left(\frac{m-1}{m+2}\right)\right]\int_0^1 \mathrm{d}r\, p^2(r) \\
&+\frac{1}{2}h^2\left[\left(\int_0^1 \mathrm{d}r\, p(r)\right)^2+\int_0^1 \mathrm{d}r\, p^2(r)\right]-\frac{1}{4}m(m-1)\left(\tau+\frac{2}{(m+2)}\right)x^2 \\
&+\frac{1}{6}(m-1)\int_0^1 \mathrm{d}r\left(rq^3(r)+3q(r)\int_0^r \mathrm{d}w\, q^2(w)\right) \\
&+\frac{1}{6}\int_0^1 \mathrm{d}r\left(rp^3(r)+3p(r)\int_0^r \mathrm{d}w\, p^2(r)\right) \\
&-\frac{1}{2}\frac{m(m-1)}{(m+2)}\left(\int_0^1 \mathrm{d}r\,(p^2(r)-q^2(r))\right)x \\
&-\frac{1}{8}\left[\int_0^1 \mathrm{d}r\left((r^2+1)p^4(r)+4p^2(r)\int_0^r \mathrm{d}y\, p^2(y)+4p(r)\int_0^r \mathrm{d}y\, yp^3(y)\right.\right. \\
&\left.\left.+12p(r)\int_0^r \mathrm{d}y\, p(y)\int_0^y \mathrm{d}z\, p^2(z)\right)\right] \\
&-\frac{1}{8}(m-1)\left[\int_0^1 \mathrm{d}r\left((r^2+1)q^4(r)+4q^2(r)\int_0^r \mathrm{d}y\, q^2(y)+4q(r)\int_0^r \mathrm{d}y\, yq^3(y)\right.\right. \\
&\left.\left.+12q(r)\int_0^r \mathrm{d}y\, q(y)\int_0^y \mathrm{d}z\, q^2(z)\right)\right] \\
&-\frac{1}{8}\frac{(m^2-2m-2)}{(m+2)^2}\int_0^1 \mathrm{d}r\, p^4(r)+\frac{1}{8}\frac{(m-1)(m^2-2)}{(m+2)^2}\int_0^1 \mathrm{d}r\, q^4(r) \\
&+\frac{1}{4}\frac{m-1}{m+2}\left[\left(\int_0^1 \mathrm{d}r\, p^2(r)\right)^2+\int_0^1 \mathrm{d}r\, p^4(r)+\int_0^1 \mathrm{d}r\, q^4(r)+\left(\int_0^1 \mathrm{d}r\, q^2(r)\right)^2\right] \\
&-\frac{1}{2}\frac{m-1}{m+2}\left[\int_0^1 \mathrm{d}r\, p^2(r)q^2(r)+\left(\int_0^1 \mathrm{d}r\, p^2(r)\right)\left(\int_0^1 \mathrm{d}r\, q^2(r)\right)\right] \\
&+\mathrm{O}(p^5, q^5, x^3, \ldots). \qquad (7)
\end{aligned}
$$

We have absorbed a factor T^{-2} into q, p, x and $\tau = T^2-1$, $h^2 = H^2/T^2$.

At the extrema the uninteresting quadrupolar order parameter x can be eliminated to give

$$
\begin{aligned}
0=\frac{\partial F}{\partial p(r)}=&-\tfrac{1}{2}h^2\left(1-2\int_0^1 \mathrm{d}r\, p(r)\right)+\frac{1}{2}\left(\tau-\frac{m(m+1)}{m+2}h^2\right)p(r)+\tfrac{1}{2}rp^2(r) \\
&+\frac{1}{2}\int_0^r \mathrm{d}y\, p^2(y)+p(r)\int_r^1 \mathrm{d}y\, p(y)-\frac{1}{2}\left[(r^2+1)p^3(r)+2p(r)\int_0^1 \mathrm{d}y\, p^2(y)\right. \\
&+\int_0^r \mathrm{d}y\, yp^3(y)+3rp^2(r)\int_r^1 \mathrm{d}y\, p(y)+3\int_0^r \mathrm{d}y\, p(y)\int_0^y \mathrm{d}z\, p^2(z) \\
&\left.+3\left(\int_r^1 \mathrm{d}y\, p(y)\right)\left(\int_0^r \mathrm{d}y\, p^2(y)\right)+3p(r)\int_r^1 \mathrm{d}y\, p(y)\int_r^y \mathrm{d}z\, p(z)\right]
\end{aligned}
$$

$$-\frac{1}{2}\frac{(m^2-2m-2)}{(m+2)^2}p^3(r)+\frac{m-1}{m+2}p(r)(p^2(r)-q^2(r))$$
$$+\tfrac{1}{2}(m-1)p(r)\int_0^1 \mathrm{d}r(p^2(r)-q^2(r))+\dots \tag{8}$$

$$0=\frac{\partial F}{\partial q(r)}=\tfrac{1}{2}(m-1)(\tau+h^2)q(r)+(m-1)\Big(\tfrac{1}{2}rq^2(r)+\frac{1}{2}\int_0^r \mathrm{d}y\,q^2(y)+q(r)\int_r^1 \mathrm{d}y\,q(y)\Big)$$
$$+\tfrac{1}{2}(m-1)\Big[(r^2+1)q^3(r)+2q(r)\int_0^1 \mathrm{d}y\,q^2(y)$$
$$+\int_0^r \mathrm{d}y\,yq^3(y)+3rq^2(r)\int_r^1 \mathrm{d}y\,q(y)+3\int_0^r \mathrm{d}y\,q(y)\int_0^y \mathrm{d}z\,q^2(z)$$
$$+3\Big(\int_r^1 \mathrm{d}y\,q(y)\Big)\Big(\int_0^r \mathrm{d}y\,q^2(y)\Big)+3q(r)\int_r^1 \mathrm{d}y\,q(y)\int_r^y \mathrm{d}y\,q(y)\Big]$$
$$+\frac{1}{2}\frac{(m-1)(m^2-2)}{(m+2)^2}q^3(r)+\Big(\frac{m-1}{m+2}\Big)q(r)(q^2(r)-p^2(r))$$
$$+\tfrac{1}{2}(m-1)q(r)\int_0^1 \mathrm{d}y(q^2(y)-p^2(y))+\dots. \tag{9}$$

To solve these equations it is useful to follow Parisi and generate the following integro-differential equations by direct differentiation with respect to r,

$$A(p)\frac{\partial p}{\partial r}-C(p,q)\frac{\partial q}{\partial r}=0 \tag{10}$$

$$C(p,q)\frac{\partial p}{\partial r}-B(q)\frac{\partial q}{\partial r}=0 \tag{11}$$

where to $\mathrm{O}(p^2, q^2, pq, \dots)$ the characteristic functions A, B, C are given by

$$A=\frac{1}{2}\frac{h^2}{p(r)}\Big(1-2\int_0^1 \mathrm{d}y\,p(y)\Big)+\frac{1}{2p(r)}\Big(rp^2(r)-\int_0^r \mathrm{d}y\,p^2(y)\Big)+\frac{(m^2+4m-2)}{(m+2)^2}p^2(y)$$
$$-\frac{1}{2p(r)}\Big[2(r^2+1)p^3(r)-\int_0^r \mathrm{d}y\Big(yp^3(y)-3rp^2(r)p(y)$$
$$+3p(y)\int_0^y \mathrm{d}w\,p^2(w)+3p^2(y)\int_r^1 \mathrm{d}w\,p(w)\Big)\Big]+\dots \tag{12}$$

$$B=\frac{1}{2q(r)}\Big(rq^2(r)-\int_0^r \mathrm{d}y\,q^2(y)\Big)+\frac{(m^2+2m+2)}{(m+2)^2}q^2(y)$$
$$-\frac{1}{2q(r)}\Big[2(r^2+1)q^3(r)-\int_0^r \mathrm{d}y\Big(yq^3(y)-3rq^2(r)q(y)$$
$$+3q(y)\int_0^y \mathrm{d}w\,q^2(w)+3q^2(y)\int_r^1 \mathrm{d}w\,p(w)\Big)\Big]+\dots \tag{13}$$

$$C = 2pq\frac{(m-1)}{(m+2)^2} + \ldots. \tag{14}$$

As in the Ising case the functions A, B, C are closely related to the stability matrix S in the restricted space of order parameters $p(r)$, $q(r)$ (Thouless *et al* 1980)

$$\begin{aligned} S &\equiv \begin{bmatrix} \dfrac{\partial^2 F}{\partial p(r)\,\partial p(s)} & \dfrac{\partial^2}{\partial p(r)\,\partial q(s)} \\ \dfrac{\partial^2 F}{\partial q(r)\,\partial p(s)} & \dfrac{\partial^2 F}{\partial q(r)\,\partial q(s)} \end{bmatrix} \\ &= \delta(r-s)\begin{bmatrix} A(p) & C(p,q) \\ C(p,q) & B(q) \end{bmatrix} \\ &\quad + \begin{bmatrix} \theta(r-s)p(s)+\theta(s-r)p(r) & 2\dfrac{m-1}{m+2}p(r)q(s) \\ 2\dfrac{m-1}{m+2}q(r)p(s) & \theta(r-s)q(s)+\theta(s-r)q(r) \end{bmatrix} \\ \theta(x) &\equiv \begin{cases} 1 & x>0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{15}$$

Stability requires that all the eigenvalues of S are non-negative.

Let us now discuss the solutions for $p(r)$, $q(r)$. As τ is lowered progressively from high temperatures we find (1) $\tau > \tau_{GT}(h)$; the phase is paramagnetic; $q=0$, $p(r) =$ constant ($= h/\sqrt{2}$ for $\tau \ll h$). (2) $h^{2/3} \gg |\tau| \gg \tau_{GT}(h)$; at the transverse spin glass transition the stability matrix S softens in the direction controlled principally by $B(q)$, leading to RS broken solutions of the form of figure 1 which are smoothly modified as τ is reduced until in the regime $h^{2/3} \gg |\tau| \gg h$ the large r behaviour is isotropic (figure 2).

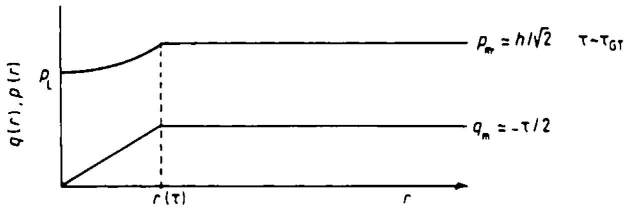

Figure 1.

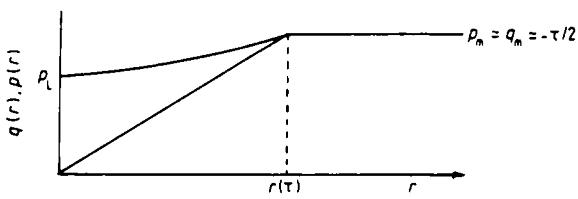

Figure 2.

Throughout the region $|\tau| \ll h^{2/3}$ RS breaking is dominantly transverse and to leading order ($\tau_{GT} \equiv 1$)

$$q_m = -\tau/2 \tag{16}$$

$$p_m = \tfrac{1}{4}\{[(\tau^2 + 8h^2)]^{1/2} - \tau\} \tag{17}$$

$$p_L = p_m\left(1 - \frac{1}{2}\frac{m-1}{(m+2)^2}\frac{p_M\tau^2}{h^2}\right) \simeq p_M \tag{18}$$

and

$$q(r) = \frac{1}{6}\frac{(m+2)^2}{(m+1)}r + O(r^2) \qquad \frac{p(r) - p_L}{p(r)p_L} = 2\frac{(m-1)}{(m+2)^2}\left(\frac{q(r)}{h}\right)^2 + O(r^3). \tag{19}$$

(3) $|\tau| > h^{2/3}$; as $|\tau|$ passes through the $h^{2/3}$ domain the fluctuations controlled principally by $A(p)$ become of dominant importance, leading to change in the analytic character of the solutions. In particular near the origin $r = 0$ we observe that for stability

$$p_L^3 \leqslant \tfrac{1}{12}(m+2)^2h^2 \tag{20}$$

whence, as τ increases further, p_L is effectively pinned, as in the Ising case. For the Ising system this bound is saturated leading to $p_L = p^*$ for all $|\tau| \gg h^{2/3}$; here we make the ansatz

$$(m-1)\lim_{r\to 0}(qq'/p') = 0 \tag{21}$$

in order to decouple (10) at the origin and retain this feature. A complete description of the behaviour near $|\tau| \simeq h^{2/3}$ has to date proved illusive†, however, qualitatively we find the position shown in figure 3, where now additionally for $r(\tau) > r \gg h^{2/3}$ the solution is isotropic; $q(r) = p(r) = \tfrac{1}{6}(m+2)r + O(r^2)$. Finally, therefore, for $|\tau| \gg h^{2/3}$ the system is essentially fully isotropic as shown in figure 4 with effects due to the magnetic field confined to a small domain $O(h^{2/3})$ at the origin.

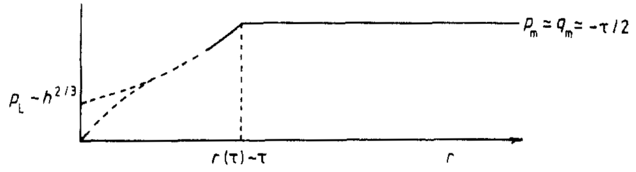

Figure 3.

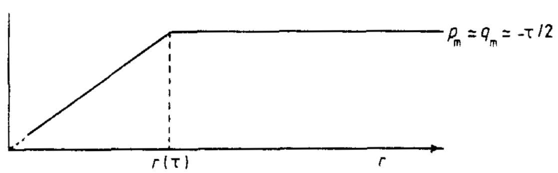

Figure 4.

† Equations (10), (11) place strong constraints on the behaviour at $r = 0$; in particular the simplest solution $p = \text{constant} + r^a$, $q = r^b$ is unacceptable for all a, b if $|\tau| \gg h^{2/3}$.

Due to the strong coupling of transverse and longitudinal aspects for $|\tau|>h^{2/3}$ explicit calculations will demand extensive numerical work except where the decoupling between thermal and magnetic effects can be put to use. For example, in the difficult $|\tau|>h^{2/3}$ regime the longitudinal local susceptibility χ_{L}^{l} is given by (5), (8) as

$$\chi_{\mathrm{L}}^{l}=1-\frac{3}{(m+2)^2}p_{\mathrm{L}}^2-\frac{h^2}{2p_{\mathrm{L}}}+\ldots \tag{22}$$

where we would expect p_{L} of order $h^{2/3}$ and relatively insensitive to the temperature. If the ansatz (21) is confirmed, as seems likely, we obtain

$$\chi_{\mathrm{L}}^{l}=1-3\left(\frac{3}{16}\right)^{1/3}\frac{1}{(m+2)^{2/3}}h^{4/3}+\ldots \tag{23}$$

which is strikingly different from the linear response expression

$$\chi_{\mathrm{L}}^{l}(\mathrm{LR})=1-\frac{3}{(m+2)}\frac{\tau^2}{4}+\mathrm{O}(\tau^3). \tag{24}$$

For $|\tau|\ll h^{2/3}$, where RS breaking occurs predominantly in the transverse modes,

$$\begin{aligned}\chi_{\mathrm{T}}^{l}&=1-p_{\mathrm{m}}^2/(m+2)+\ldots\\ \chi_{\mathrm{L}}^{l}&=1-\tfrac{1}{4}(\tau+\sqrt{\tau^2+8h^2})\end{aligned} \tag{25}$$

where at this order $\tau_{\mathrm{GT}}=1$. At scales $|\tau|\gg h$ the local susceptibility is therefore essentially isotropic

$$\chi_{\mathrm{T}}^{l}=\chi_{\mathrm{L}}^{l}+\mathrm{O}(\tau^2).$$

More importantly, for $|\tau|\ll h^{2/3}$ we should expect irreversibility to be markedly greater in the transverse rather than longitudinal direction since

$$\begin{aligned}\chi_{\mathrm{T}}^{l}-\chi_{\mathrm{T}}^{l}(\mathrm{LR})&=3\frac{(m+1)}{(m+2)^2}(\tau/2)^2+\ldots\\ \chi_{\mathrm{L}}^{l}-\chi_{\mathrm{L}}^{l}(\mathrm{LR})&=\mathrm{O}(p_m(\tau^3/h^2)).\end{aligned} \tag{26}$$

Finally, some comments about the thermodynamic instability of our solutions are in order. Returning to (10), (11) we see that consistency demands

$$AB=C^2 \tag{27}$$

whence in the stability matrix S there are modes for which the local contribution vanishes identically, leaving

$$S=\begin{bmatrix}\theta(r-s)p(s)+\theta(s-r)p(r) & 2\left(\frac{m-1}{m+2}\right)p(r)q(s)\\ 2\left(\frac{m-1}{m+2}\right)q(r)p(s) & \theta(r-s)q(s)+\theta(s-r)q(r)\end{bmatrix}.$$

Extending the work of Thouless *et al* (1980) we should thus expect to observe a marginal stability, although only for $|\tau|\ll h^{2/3}$ has this been checked.

A related study has been carried out independently and simultaneously by Gabay *et al* (1982). Where they overlap, our results are in broad agreement.

Financial support from the SERC is gratefully acknowledged.

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